

The Epsilon Expansion of Feynman Diagrams via Hypergeometric Functions and Differential Reduction

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Higher-order diagrams required for radiative corrections to mixed electroweak and QCD processes at the LHC and anticipated future colliders will require numerically stable representations of the associated Feynman diagrams. The hypergeometric representation supplies an analytic framework that is useful for deriving such stable representations. We discuss the reduction of Feynman diagrams to master integrals, and compare integration-by-parts methods to differential reduction of hypergeometric functions. We describe the problem of constructing higher-order terms in the epsilon expansion, and characterize the functions generated in such expansions.

1. Introduction

A variety of approaches are known for evaluating Feynman diagrams analytically [1] and constructing the ε expansion of dimensionally-regularized integrals. [2] One-loop diagrams [3–6] are expressible in terms of generalized hypergeometric functions of the form ${}_pF_p$, Appell functions F_1, F_2, F_3, F_4 , or Horn-type [7] multivariable functions. [8] Analytical techniques for evaluating the finite part of one-loop diagrams have long been known, [9] but higher order diagrams require more terms in the ε expansion. These facts motivate us to seek a way to obtain the all-order ε expansion of Horn-type hypergeometric functions in general.

The first such construction of the all-order ε expansion was obtained for the Gauss hypergeometric function with special, physically-motivated, values of the parameters. [10] The first systematic algorithm for constructing the ε expansion of multi-variable hypergeometric functions was described in Ref. [12], using Goncharov's polylogarithms [11] in the case with integer values of parameters. However, diagrams with massive propagators lead to hypergeometric functions with rational values of parameters. [13] This case generates multiple (inverse) binomial sums which were first systematically investigated in Refs. [14–16]. Ref. [17] generalized the technique of Ref. [12] to the case of rational parameters, but the method was limited to the “zero-balanced” case, and applied to ${}_pF_p$ and F_1 , but not F_2 or F_3 . Some statements about the structure of the coefficients of the ε -expansion for ${}_pF_p$ with one non-balanced rational parameter contradicted explicit calculations of the first few coefficients of the ε expansion. [16, 18] In particular, the multiple sum $\sum_{j=1}^{\infty} z^j [j \binom{2j}{j}]^{-1} \sum_{k=1}^{2n-1} k^{-2}$ was evaluated analytically in Ref. [16] (see Eq. (C.22) there), and it was found that the ε expansion of the hypergeometric function

$${}_pF_p \left(\begin{matrix} I_1 + \frac{1}{2} + b_1\varepsilon, K_j + a_j\varepsilon \\ I_2 + \frac{1}{2} + f_1\varepsilon, I_3 + \frac{1}{2} + f_2\varepsilon, L_i + c_i\varepsilon \end{matrix} \middle| z \right) \quad (1)$$

with integer values of I_j, K_j, L_j leads, at weight **3**, to classical polylogarithms of argument $\frac{1-\sqrt{y}}{1+\sqrt{y}}$. (This was later confirmed in Ref. [19].) Furthermore, Ref. [18] analyzed the more general case

$${}_pF_p \left(\begin{matrix} I_1 + \frac{1}{2} + b_1\varepsilon, I_2 + \frac{1}{2} + b_2\varepsilon, K_j + a_j\varepsilon \\ I_3 + \frac{1}{2} + f_1\varepsilon, I_4 + \frac{1}{2} + f_2\varepsilon, I_5 + \frac{1}{2} + f_3\varepsilon, L_i + c_i\varepsilon \end{matrix} \middle| z \right), \quad (2)$$

via the sum $\sum_{j=1}^{\infty} z^j [j \binom{2j}{j}]^{-1} \sum_{k=1}^{2n-1} k^{-3}$, and was again found to differ from the prediction of Ref. [17].

The inadequacies of existing algorithms and a desire to explore the results of Ref. [4] for one-loop diagrams motivated the development of new technologies [20–25] and the extension of previous ones [16] for all-order expansions, [26] with the goals of

- (i) independently verifying the results of Ref. [17],
- (ii) constructing the analytical coefficients of the ε expansion of hypergeometric functions of several variables.

Recently, new generalizations of the algorithm of Ref. [12] have been found using cyclotomic harmonic sums [27] and Z -sums, [28] allowing the ε expansion to be constructed in the cases of Eqs. (1) and (2). Their applicability to multivariable hypergeometric functions is not yet clear.

The higher terms in the ε expansion of one-loop diagrams can be constructed [9] without applying algorithms for expanding hypergeometric functions. [29–32] However, results derived via the hypergeometric representation are more compact and often can be expressed in terms of simpler functions, *e.g.* Nielsen’s polylogarithms instead of Goncharov’s. This can be seen, for example, by comparing the results presented in Refs. [29, 30] to those in Refs. [10, 16, 33]. Some remarkable results on hexagon diagrams were presented in Ref. [32], but this was done via an experimental mathematical analysis [34] specialized to this case. Methods following from the properties of hypergeometric functions hold the promise of being applicable to wide classes of Feynman diagrams. In addition, the ε expansion of hypergeometric functions has interesting applications beyond Feynman diagrams, in a broader mathematical context. [35]

In this paper, we will concentrate on Horn-type hypergeometric functions, a very general class which encompasses all of those mentioned earlier, and which we conjecture [36] to be general enough to express all Feynman diagrams. Section 2 describes the differential reduction of this class of hypergeometric functions. Section 3 describes the differential reduction in application to Feynman diagrams and explain the difference in counting master integrals via our differential technologies [37] versus integration-by-parts techniques [38]. Section 4 describes our approach to constructing the ε expansion, and provides some examples.

2. Differential Reduction of Horn-type Hypergeometric Functions

Let us begin by considering the generalized hypergeometric functions ${}_{p+1}F_p(\mathbf{a}; \mathbf{b}; z)$ defined by a series about $z = 0$ as

$${}_{p+1}F_p(\mathbf{a}; \mathbf{b}; z) = \sum_{j=0}^{\infty} \frac{\prod_{i=1}^{p+1} (a_i)_j}{\prod_{k=1}^p (b_k)_j} \frac{z^j}{j!}, \quad (3)$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is the Pochhammer symbol, and $\mathbf{a} = (a_1, \dots, a_p)$, $\mathbf{b} = (b_1, \dots, b_q)$ are called the upper and lower parameters, respectively. This function satisfies a differential equation of order $p+1$, so that there are $p+1$ independent solutions for unexceptional values of the parameters.

“Differential reduction” is an algorithm by which any function ${}_{p+1}F_p(\mathbf{a} + \mathbf{m}; \mathbf{b} + \mathbf{k}; z)$ with integer parameter lists \mathbf{m}, \mathbf{k} may be expressed in terms of a set of p functions in which the values of these arguments are shifted by lists of integers $(\mathbf{l})_j$ and $(\mathbf{r})_j$, respectively, with $j = 1, \dots, p$. This reduction has the form

$$Q_{p+1}(\mathbf{a}, \mathbf{b}, z) {}_{p+1}F_p(\mathbf{a} + \mathbf{m}; \mathbf{b} + \mathbf{k}; z) = \sum_{j=0}^p Q_j(\mathbf{a}, \mathbf{b}, z) {}_{p+1}F_p(\mathbf{a} + \mathbf{l}_j; \mathbf{b} + \mathbf{r}_j; z) \quad (4)$$

where the Q_j are a set of polynomials in a_i, b_i , and z . The resulting expression can be converted into derivatives of the unshifted hypergeometric function, so that the reduction takes the form [39]

$$S(\mathbf{a}, \mathbf{b}, z) {}_{p+1}F_p(\mathbf{a} + \mathbf{m}; \mathbf{b} + \mathbf{k}; z) = \sum_{j=0}^p R_j(\mathbf{a}, \mathbf{b}, z) \theta^j {}_{p+1}F_p(\mathbf{a}; \mathbf{b}; z), \quad (5)$$

where $\theta = z d/dz$ and S and R_i are other polynomials in a_i, b_i , and z .

For particular values of the parameters, an algebraic solution of the corresponding differential equations can be found, so that the differential reduction simplifies. For example, when some of the upper parameters are integers I_i , we have [40]

$$\tilde{P}(\mathbf{a}, \mathbf{b}, z) {}_{p+1}F_p(\mathbf{I}, \mathbf{a} + \mathbf{m}; \mathbf{b} + \mathbf{k}; z) = \sum_{j=0}^{p-1} \tilde{R}_j(\mathbf{a}, \mathbf{b}, z) \theta^j {}_{p+1}F_p(\mathbf{I}, \mathbf{a}; \mathbf{b}; z) + \tilde{R}_p(\mathbf{a}, \mathbf{b}, z), \quad (6)$$

with algebraic functions $\tilde{R}_p(\mathbf{a}, \mathbf{b}, z)$. In this case, the Eq. (4) has the form

$$\tilde{Q}_{p+1}(\mathbf{a}, \mathbf{b}, z) {}_{p+1}F_p(\mathbf{I}, \mathbf{a} + \mathbf{m}; \mathbf{b} + \mathbf{k}; z) = \sum_{j=0}^{p-1} \tilde{Q}_j(\mathbf{a}, \mathbf{b}, z) {}_{p+1}F_p(\mathbf{J}, \mathbf{a} + \mathbf{l}_j; \mathbf{b} + \mathbf{r}_j; z) + \tilde{R}(\mathbf{a}; \mathbf{b}; z), \quad (7)$$

where \mathbf{I}, \mathbf{J} are lists of integer parameters.

This reduction can be generalized to all multivariable Horn-type hypergeometric functions [39] having L independent solutions:

- (i) For unexceptional values of parameters (irreducible monodromy group), linear differential relations can be found among $L + 1$ functions with parameters differing by integers.
- (ii) When the monodromy group is reducible, the number of functions entering into these differential relations is correspondingly reduced.

In the reducible case, a basis can currently be constructed only when the differential reduction has been carried out explicitly.

3. Counting the Number of Basis Elements

3.1. Preliminary Considerations

Consider a function $f(z)$ satisfying a homogeneous differential equation of order k with polynomial coefficients:

$$\sum_{j=0}^k a_j(z) \left(\frac{d}{dz} \right)^j f(z) = 0. \quad (8)$$

It is easily shown that the function $H(z) = z^a f(z)$ satisfies a differential equation of the same order as $f(z)$. Consider a function $H(z)$ which is a linear combination of functions $f_j(z)$ satisfying homogeneous differential equations of order r_k with rational coefficients,

$$H(z) = \sum_{k=1}^m c_k x^{\alpha_k} f_k(z), \quad (9)$$

$$\left(\frac{d}{dz} \right)^{r_k} f_k(z) + \sum_{j=0}^{r_k-1} a_j(z) \left(\frac{d}{dz} \right)^j f_k(z) = 0, \quad (10)$$

where α_k are rational numbers and r_k are integers. Then it can be shown that the function $H(z)$ satisfies a differential equation with rational coefficients of order not less than $m \times r_k$.

We can take the f_k to be Horn-type hypergeometric functions (multi-variable, in general) whose parameters are linear combinations of the parameters of some Horn-type function $H(\mathbf{J}; \mathbf{z})$, which plays the role of $H(z)$ above. Then, in accordance with Sec. 2, there are linear differential relations between any $m \times r_k + 1$ of the functions $H(\mathbf{J} + \mathbf{m}; \mathbf{z})$. This means that these functions can be expressed in terms of a set of basis functions shifted by integer values of the parameters,

$$H(\mathbf{J} + \mathbf{m}; \mathbf{z}) = \sum_{K=0}^L R_K(\mathbf{z}) \frac{\partial^K}{\partial z_1^{k_1} \dots \partial z_r^{k_r}} H_i(\mathbf{J}; \mathbf{z}). \quad (11)$$

The number L of elements on the r.h.s. of this relation is equal to the number of solutions of the corresponding differential equation.

3.2. Application to Feynman Diagrams

As an application of the reduction in Sec. 2, a series of publications [40–43] has analyzed the Feynman diagrams $\Phi(n, \mathbf{j}; z)$ having a Mellin-Barnes representation of the form

$$\Phi(n, \mathbf{j}; z) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} dt (\kappa z)^t \Gamma(-t) F(t), \quad (12)$$

where \mathbf{j} is the list of powers of the propagators in the Feynman diagram, n is the dimension of space-time, κ is a constant, and $F(t)$ has the structure

$$F(t) = \prod_{i,j,k,l} \frac{\Gamma(A_i + t) \Gamma(C_k - t)}{\Gamma(B_j + t) \Gamma(D_l - t)}, \quad (13)$$

where A_i, B_j, C_k, D_l are linear combinations of n and \mathbf{j} of the form $A_i = a_0 n + \sum_k a_k j_k$, etc., and the dimensions of the lists $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ of these linear combinations satisfy the condition

$$\dim \mathbf{A} + \dim \mathbf{D} - \dim \mathbf{B} - \dim \mathbf{C} = 1. \quad (14)$$

Closing the integration contour gives

$$\Phi(n, \mathbf{j}; z) = \sum_{a=1}^q z^{l_a} C_{l_a}(n, \mathbf{j}) {}_{p+1}F_p(\tilde{\mathbf{A}}_a; \tilde{\mathbf{B}}_a; \kappa z), \quad (15)$$

where q is an integer, $l_a, \tilde{\mathbf{A}}_a, \tilde{\mathbf{B}}_a$ are linear combinations of n and \mathbf{j} with rational coefficients, $z \neq 1$, and the coefficients C_{l_a} are products of Γ -functions with arguments depending only on n and \mathbf{j} .

In accordance with the preliminary remarks in this section, the maximal number of basis elements for Eq. (15) is equal to $q \times p$. This number should coincide with the number of master integrals found by other means, in particular, by integration-by-parts (IBP) relations [38]. However, some of the parameters of the hypergeometric functions on the r.h.s. of Eq. (15) may be exceptional. Typically, this occurs when the upper parameters are integers, or the difference between an upper and lower parameter is a positive integer. [40] In these cases, the actual number of basis elements will be less than the typical number. (See also the discussion in Ref. [42].)

The number of nontrivial basis elements for a hypergeometric function of one variable may be defined to be the highest power of the differential operator $\theta = z d/dz$ in its differential reduction: [40]

$${}_{p+1}F_p(\mathbf{A}; \mathbf{B}; z) = \sum_s \left[\sum_{l=0}^v P_l(z) \theta^l {}_{s+1}F_s(\mathbf{A} - \mathbf{I}_1; \mathbf{B} - \mathbf{I}_2; z) + R_s(z) \right], \quad (16)$$

where $\mathbf{I}_1, \mathbf{I}_2$ are lists of integers and $P_l(z), R(z)$ are rational functions. On the other hand, applying the IBP relations to Eq. (15) would lead to an expression of the form

$$\Phi(n, \mathbf{j}; z) = \sum_{k=1}^h B_k(n, \mathbf{j}; z) \Phi_k(n; z), \quad (17)$$

where some of the master integrals Φ_k are expressible solely in terms of Γ -functions.[44]

Analyzing the relation between the reductions (16) and (17) leads to the following criteria for determining the number of master integrals:¹

- (i) Each term in the hypergeometric representation of a Feynman diagram Eq. (15) has the same number L of nontrivial basis elements (up to rational functions).
- (ii) The number of master integrals following from the IBP relations which are not expressible solely in terms of Γ -functions [44] is equal to the number L of nontrivial basis elements.

3.3. Examples: Vertex-Type Diagrams

In this section, we consider two examples of the differential reduction of one-loop vertex-type diagrams.

Example I: Let us consider the one-loop vertex (C_3 in the notation of Ref. [40]; see Eq. (52) there for details):

$$\begin{aligned} C_3^{(q)}(j_1, j_2, \sigma) &\equiv \int \frac{d^n k}{[(k-p_1)^2 - m^2]^{j_1} [(k+p_2)^2 - m^2]^{j_2} (k^2)^\sigma} \Big|_{p_1^2=p_2^2=0} \\ &= i^{1-n} \pi^{n/2} (-m^2)^{\frac{n}{2}-\sigma-j_{12}} \frac{\Gamma(j_{12} + \sigma - \frac{n}{2}) \Gamma(\frac{n}{2} - \sigma)}{\Gamma(j_{12}) \Gamma(\frac{n}{2})} {}_4F_3 \left(j_{12} + \sigma - \frac{n}{2}, j_1, j_2, \frac{n}{2} - \sigma \mid \frac{(p_1 - p_2)^2}{4m^2} \right), \end{aligned} \quad (18)$$

where $j_{12} \equiv j_1 + j_2$, $\sigma \equiv \sum_{k=1}^{q+1} \sigma_k$, and the “dressed” massless propagator is

$$\frac{1}{(k^2)^\sigma} \equiv \left\{ \prod_{k=1}^{q+1} \frac{\Gamma(\frac{n}{2} - \sigma_k)}{\Gamma(\sigma_k)} \right\} \frac{[i^{1-n} \pi^{n/2}]^q \Gamma(\sigma - \frac{n}{2} q)}{\Gamma(\frac{n}{2}(q+1) - \sigma)} \frac{1}{(k^2)^{\sigma - \frac{n}{2} q}}, \quad (19)$$

¹See Ref. [40] for details. We thank Bas Tausk for bringing to our attention the fact that the diagram F_2 in that paper was analyzed in detail in Refs. [48] (Eqs. (74), (75)) and [49] (Eqs. (124), (130)). The results of these analyses are in agreement.

where q is the number of massless loops, $(q+1)$ is the number of massless lines, and $q=0$ corresponds to a massless line without additional internal loops. ($C_3^{(0)}$ is a true one-loop vertex.)

With the redefinition $\sigma \rightarrow \sigma - \frac{n}{2}q$, the hypergeometric function in Eq. (18) takes the form

$${}_4F_3 \left(j + \sigma - \frac{n}{2}(q+1), j_1, j_2, \frac{n}{2}(q+1) - \sigma \left| \frac{(p_1 - p_2)^2}{4m^2} \right. \right). \quad (20)$$

In accordance with the differential reduction algorithm, this function may be expressed in terms of a ${}_3F_2$ function with one unit upper parameter, and its first derivative,

$$\{1, \theta\} \times {}_3F_2 \left(1, \frac{n}{2}(q+1) + I_1, I_2 - \frac{n}{2}(q+1) \left| \frac{n}{2}, \frac{1}{2} + I_3 \right. \right| z \right), \quad (21)$$

together with rational functions of z , for integer values I_1, I_2, I_3 . The short-hand notation $(1, \theta)$ stands for a combination $P_1(z) + P_2(z)\theta$, with rational functions P_i . (See Eqs. (17) and (20) in Ref. [40].) According to the criteria in Sec. 3.2, there are two master integrals for this diagram which are not expressible in terms of Γ functions. For $q=1$, the standard approach based on IBP relations, [38] yields two two-loop vertex master integrals. (See Eqs. (3) and (9) in Ref. [50].) These master integrals are relevant for the massless fermion contribution to Higgs production and decay. [51] It is interesting to note that at the one-loop level, $C_3^{(0)}$ has one master integral of the vertex type and one of the propagator type, which are again equivalent to Eq. (21).

Example II: Let us consider one-loop vertex diagram $C_1^{(q_1, q_2)}$, defined as

$$C_1^{(q_1, q_2)}(\sigma_1, \sigma_2, \rho) \equiv \int \frac{d^n k}{[(k-p_1)^2]^{\sigma_1} [(k+p_2)^2]^{\sigma_2} (k^2 - m^2)^\rho} \Big|_{p_1^2 = p_2^2 = 0}, \quad (22)$$

where σ_1, σ_2 are defined as in Eq. (19), with $\sigma_j = \sum_{k=1}^{q_j+1} \sigma_{jk}$ for $j=1, 2$. The case $q_1=0, q_2=1$ corresponds to Eq. (173) in Ref. [52]. The hypergeometric representation for this diagram is [53]

$$\begin{aligned} C_1^{(q_1, q_2)}(\sigma_1, \sigma_2, \rho) &= i^{1-n} \pi^{n/2} (-m^2)^{\frac{n}{2} - \rho - \sigma_1 - \sigma_2} \\ &\left\{ \frac{\Gamma(\rho + \sigma_1 + \sigma_2 - \frac{n}{2}) \Gamma(\frac{n}{2} - \sigma_1 - \sigma_2)}{\Gamma(\frac{n}{2}) \Gamma(\rho)} {}_3F_2 \left(\rho + \sigma_1 + \sigma_2 - \frac{n}{2}, \sigma_1, \sigma_2 \left| \frac{n}{2}, 1 + \sigma_1 + \sigma_2 - \frac{n}{2} \right. \right) - \frac{(p_1 - p_2)^2}{m^2} \right\} \\ &+ \left(-\frac{(p_1 - p_2)^2}{m^2} \right)^{\frac{n}{2} - \sigma_1 - \sigma_2} \frac{\Gamma(\frac{n}{2} - \sigma_1) \Gamma(\frac{n}{2} - \sigma_2) \Gamma(\sigma_1 + \sigma_2 - \frac{n}{2})}{\Gamma(n - \sigma_1 - \sigma_2) \Gamma(\sigma_1) \Gamma(\sigma_2)} \\ &\times {}_3F_2 \left(\rho, \frac{n}{2} - \sigma_1, \frac{n}{2} - \sigma_2 \left| n - \sigma_1 - \sigma_2, \frac{n}{2} - \sigma_1 - \sigma_2 + 1 \right. \right) - \frac{(p_1 - p_2)^2}{m^2} \Big\}. \end{aligned} \quad (23)$$

In accordance with the differential reduction algorithm, the first ${}_3F_2$ function in Eq. (24) can be expressed in terms of ${}_2F_1$, and the second one can be expressed in terms of ${}_3F_2$ with one unit upper parameter, namely

$$\{1, \theta\} \times {}_2F_1 \left(\sigma_1, \sigma_2 \left| \frac{n}{2} \right. \right| z \right), \quad \{1, \theta\} \times {}_3F_2 \left(1, \frac{n}{2} - \sigma_1, \frac{n}{2} - \sigma_2 \left| n - \sigma_1 - \sigma_2, \frac{n}{2} - \sigma_1 - \sigma_2 + 1 \right. \right| z \right), \quad (24)$$

together with rational functions of z . We note that each hypergeometric function in Eq. 24 has the same number of nontrivial basis elements, in agreement with criterion (i) in section 3.2. When one (or both) of the σ_j are integers ($q_j=0$), further reduction is possible to

$${}_2F_1 \left(1, \sigma_2 \left| \frac{n}{2} \right. \right| z \right), \quad {}_2F_1 \left(1, \frac{n}{2} - \sigma_2 \left| n - \sigma_1 - \sigma_2 \right. \right| z \right). \quad (25)$$

In accordance with our criteria, there are two nontrivial master integrals for non-integer σ_j , and one nontrivial master integral when one of the σ_j is an integer. The last result in agreement with Ref. [52].

3.4. Counting the Non-Trivial Master Integrals

When analyzing various diagrams, one example was found [37] where our criteria allow us to predict and prove an additional relation between master integrals. It is a two-loop sunset diagram relevant for the evaluation of $\mathcal{O}(\alpha_s)$ relation between the pole and $\overline{\text{MS}}$ mass of the top-quark in the Standard Model, [54] as well as in the gaugeless-limit model. [55] To illustrate this, let us consider the two-loop self-energy diagram

$$V_{1200}(\rho, \sigma, \alpha, \beta, m^2, M^2) = \int \frac{d^n(k_1 k_2)}{[(k_1 - p)^2 - m^2]^\rho [(k_1 - k_2)^2 - M^2]^\sigma [k_2^2]^\alpha [k_1^2]^\beta} \Big|_{p^2 = m^2}, \quad (26)$$

which is generated in an intermediate step of the calculations in Ref. [55]. The $\alpha = 0$ case corresponds to a diagram considered in Ref. [37]. The Mellin-Barnes representation of the integral (26) is

$$V_{1200}(\rho, \sigma, \alpha, \beta, m^2, M^2) = [i^{1-n} \pi^{\frac{n}{2}}]^2 \frac{\Gamma(\frac{n}{2} - \alpha)}{\Gamma(\alpha) \Gamma(\rho) \Gamma(\sigma)} (-m^2)^{n-\alpha-\beta-\sigma-\rho} \int ds \left(\frac{M^2}{m^2} \right)^s \frac{\Gamma(-s) \Gamma(\frac{n}{2} - \sigma - s) \Gamma(2n - 2\alpha - 2\beta - 2\sigma - \rho - 2s) \Gamma(\alpha + \beta + \sigma + \rho - n + s) \Gamma(\alpha + \sigma - \frac{n}{2} + s)}{\Gamma(n - \alpha - \sigma - s) \Gamma(\frac{3n}{2} - \alpha - \beta - \sigma - \rho - s)}. \quad (27)$$

After integration, we obtain

$$V_{1200}(\rho, \sigma, \alpha, \beta, m^2, M^2) = [i^{1-n} \pi^{\frac{n}{2}}]^2 \frac{\Gamma(\frac{n}{2} - \alpha)}{\Gamma(\alpha) \Gamma(\sigma) \Gamma(\frac{n}{2})} (-M^2)^{n-\alpha-\beta-\sigma-\rho} \times \left[\left(\frac{m^2}{M^2} \right)^{\frac{n}{2}-\beta-\rho} \frac{\Gamma(\alpha) \Gamma(\alpha + \sigma - \frac{n}{2}) \Gamma(\beta + \rho - \frac{n}{2}) \Gamma(n - 2\beta - \rho)}{\Gamma(\rho) \Gamma(n - \beta - \rho)} {}_4F_3 \left(\begin{matrix} \alpha, \alpha + \sigma - \frac{n}{2}, \frac{n-\rho}{2} - \beta, \frac{n+1-\rho}{2} - \beta \\ \frac{n}{2}, n - \beta - \rho, \frac{n}{2} + 1 - \beta - \rho \end{matrix} \middle| \frac{4m^2}{M^2} \right) + \frac{\Gamma(\alpha + \beta + \sigma + \rho - n) \Gamma(\alpha + \beta + \rho - \frac{n}{2}) \Gamma(\frac{n}{2} - \beta - \rho)}{\Gamma(\beta + \rho)} {}_4F_3 \left(\begin{matrix} \alpha + \beta + \sigma + \rho - n, \alpha + \beta + \rho - \frac{n}{2}, \frac{\rho}{2}, \frac{\rho+1}{2} \\ \frac{n}{2}, \beta + \rho, 1 + \beta + \rho - \frac{n}{2} \end{matrix} \middle| \frac{4m^2}{M^2} \right) \right]. \quad (28)$$

The differential reduction of the hypergeometric functions in Eq. (28) can be expressed as

$${}_4F_3 \left(\begin{matrix} \alpha, \alpha + \sigma - \frac{n}{2}, \frac{n-\rho}{2} - \beta, \frac{n+1-\rho}{2} - \beta \\ \frac{n}{2}, n - \beta - \rho, \frac{n}{2} + 1 - \beta - \rho \end{matrix} \middle| z \right) \rightarrow (1, \theta) \times {}_3F_2 \left(\begin{matrix} 1, I_1 - \frac{n}{2}, \frac{n}{2} + \frac{1}{2} + I_2 \\ n + I_3, \frac{n}{2} + I_4 \end{matrix} \middle| z \right) + R_1(z),$$

$${}_4F_3 \left(\begin{matrix} \alpha + \beta + \sigma + \rho - n, \alpha + \beta + \rho - \frac{n}{2}, \frac{\rho}{2}, \frac{\rho+1}{2} \\ \frac{n}{2}, \beta + \rho, 1 + \beta + \rho - \frac{n}{2} \end{matrix} \middle| z \right) \rightarrow (1, \theta) \times {}_3F_2 \left(\begin{matrix} I_1 - n, \frac{1}{2} + I_2, 1 \\ \frac{n}{2} + I_3, 2 \end{matrix} \middle| z \right). \quad (29)$$

In accordance with our criteria, there should exist two algebraically independent master integrals and some integrals expressible in terms of rational functions.

Diagram (26) is algebraically reducible via IBP relations [47] to four new diagrams with (i) $\sigma = 0$, (ii) $\rho = 0$, (iii) $\alpha = 0$, (iv) $\beta = 0$. The $\sigma = 0$ diagram vanishes in framework of dimensional regularization. The diagrams with $\beta = 0$ and $\alpha = 0$ are irreducible, and treated as two independent master integrals. The diagram with $\beta = 0$ is a two-loop sunset, and in accordance with the IBP algorithm of Ref. [47], it has three master integrals of the same topology plus a product of two one-loop bubbles ($\alpha = \beta = 0$). In this way, the classical IBP relations applied to diagram (26) give six master integrals, without any information about its algebraic structure. The hypergeometric representation for sunset diagrams (28) and the differential reduction algorithm (5) allow us to find algebraic relations between the three sunset-type master integrals. (See Eq.(9) in Ref. [37].) The on-shell case of a new relation (Eq.(10) in Ref. [37]) enters in the package ON-SHELL2 [56], and was postulated via a study of the higher-order coefficients of the ε -expansion in Ref. [57]. In Ref. [58], it was pointed out that the last on-shell relation can be treated purely diagrammatically. In this case, the new diagram does not follow from the original sunset by the contraction of any lines. Surprisingly, this new relation is not reproduced by the IBP algorithm of Ref. [47] or by the automated programs AIR [59] and FIRE [60]. This may be an artifact of the “topological” nature of the solution of IBP relations (systematic contractions of one line in a diagram). The effect that the solution of IBP relations does not recognize the algebraic relation between master integrals is also seen for phase-space integrals: some of the master integrals collected in Ref. [61] are also algebraically related to each other.

4. Construction of the ε Expansion

The evaluation of multi-loop radiative corrections requires higher-order terms in the ε expansion of lower-order diagrams. The integral representation [53], the series representation [12, 15–17, 26] or the differential equations approach [20, 23–25] may be used to construct the ε expansion of hypergeometric functions. We will focus on the differential equations approach. As is well-known, hypergeometric functions satisfy a differential equation

$$\left[z \Pi_{i=1}^p \left(z \frac{d}{dz} + A_i \right) - z \frac{d}{dz} \Pi_{k=1}^{p-1} \left(z \frac{d}{dz} + B_k - 1 \right) \right] {}_pF_{p-1}(\mathbf{A}; \mathbf{B}; z) = 0. \quad (30)$$

Due to the analyticity of the hypergeometric function ${}_pF_{p-1}(\mathbf{A}; \mathbf{B}; z)$ with respect to its parameters A_i, B_k , the differential equation for the coefficient functions ω_k of the Laurent expansion can be derived directly from Eq. (30) without any reference to the series or integral representation. This was the main idea of the approach developed in Refs. [20, 23, 24].

It is convenient to introduce a new parametrization, $A_i \rightarrow A_i + a_i \varepsilon, B_j \rightarrow B_j + b_j \varepsilon$, where ε is small, so that the Laurent expansion takes the form of an “ ε expansion,”

$$\omega(z) \equiv {}_pF_{p-1}(\mathbf{A} + \mathbf{a}\varepsilon; \mathbf{B} + \mathbf{b}\varepsilon; z) = {}_pF_{p-1}(\mathbf{A}; \mathbf{B}; z) + \sum_{k=1}^{\infty} \varepsilon^k \omega_k(z).$$

We can rewrite Eq. (30) as a system of first-order differential equations (called the “Pfaff form”) and expand all terms in powers of ε :

$$d\phi^{(i)}(z, \varepsilon) = \sum_{j=0}^{p-1} P_{i,j}(z, \varepsilon) \phi^{(j)}(z, \varepsilon) dz, \quad (31)$$

where $\phi^{(i)}(z) = h_i(z) \theta \phi^{(i-1)}(z), i = 1, \dots, p-1$ and $\phi^{(0)} = \omega(z)$, with arbitrary functions $h_i(z)$. The result can be written in the form

$$P_k(z) \frac{d}{dz} f_j^{(k)}(z) = \sum_{m,l} R_{j,l}^{(k,m)}(z) f_{j-1-l}^{(m)}(z), \quad j = 0, \dots, \infty, \quad (32)$$

where $P_k(z)$ and $R_{j,l}^{(k,m)}(z)$ are polynomials. For a restricted set of parameters, the expanded system of differential equations in Eq. (32) has triangular form and can be integrated iteratively in terms of Goncharov’s polylogarithms.²

For illustration, let us consider the factorization of a differential operator expanded in powers of ε :

$$D^{(p)} = \left[\Pi_{i=1}^p (\theta + A_i + a_i \varepsilon) - \frac{1}{z} \theta \Pi_{k=1}^{p-1} (\theta + B_k - 1 + b_k \varepsilon) \right] = \sum_{j=0}^p \varepsilon^j D_j^{(p-j)}(\mathbf{A}, \mathbf{B}, \mathbf{a}, \mathbf{b}, z), \quad (33)$$

where $\theta = z d/dz$, the upper index gives the order of the differential operator, $D_p^{(0)} = \Pi_{k=1}^p a_k$, and

$$\begin{aligned} D_0^{(p)} &= \Pi_{i=1}^p (\theta + A_i) - \frac{1}{z} \theta \Pi_{k=1}^{p-1} (\theta + B_k - 1) \\ &= \left\{ -(1-z) \frac{d}{dz} + \sum_{k=1}^p A_k - \frac{1}{z} \sum_{j=1}^{p-1} (B_j - 1) \right\} \theta^{p-1} + \sum_{j=1}^{p-2} \left[P_{p-j}^{(p)}(\mathbf{A}) - \frac{1}{z} P_{p-j}^{(p-1)}(\mathbf{B} - \mathbf{1}) \right] \theta^j + \sum_{i=0}^p A_i, \end{aligned} \quad (34)$$

with polynomials $P_j^{(p)}(r_1, \dots, r_p)$ defined via the relations

$$\prod_{k=1}^p (z + r_k) = \sum_{j=0}^p P_{p-j}^{(p)}(r_1, \dots, r_p) z^j \equiv \sum_{j=0}^p P_{p-j}^{(p)}(\mathbf{r}) z^j \equiv \sum_{j=0}^p P_j^{(p)}(\mathbf{r}) z^{p-j}, \quad (35)$$

²The fact that higher-order differential operators depending on ε may take a simpler form after ε -expansion has also been observed in the evaluation of multi-loop master integrals; see *e.g.* Refs. [46, 48–50, 52, 62]

and $\mathbf{1}_k \equiv 1$. As a first step toward the construction of a solution, the operator $D_0^{(p)}$ should be rewritten in the form

$$D_0^{(p)} = \left\{ -(1-z) \frac{d}{dz} + R_1 - \frac{1}{z} R_2 \right\} \Pi_{j=1}^{p-1} (\theta + \beta_j) , \quad (36)$$

where β_j are rational numbers. Eqs. (35) and (36) give rise to a system of equations

$$\begin{aligned} R_2 P_{k-1}^{p-1}(\beta) + P_k^{p-1}(\beta) &= P_k^{p-1}(\mathbf{B} - \mathbf{1}) , \quad k = 1, \dots, p . \\ R_1 P_{k-1}^{p-1}(\beta) + P_k^{p-1}(\beta) &= P_k^p(\mathbf{A}) , \quad k = 1, \dots, p . \end{aligned} \quad (37)$$

The differential equation

$$\left\{ -(1-z) \frac{d}{dz} + R_1 - \frac{1}{z} R_2 \right\} h(z) = 0 , \quad (38)$$

generates the function

$$h(z) = C z^{-R_2} (z-1)^{R_2-R_1} , \quad (39)$$

for which only three rational parametrizations are known:

$$(i) R_1 = R_2; \quad (ii) R_1 = 0; \quad (iii) R_2 = 0 . \quad (40)$$

When all $A_i = 0$ and $B_j = 1$, we have integer parameters and the differential operators are factorizable. When only one upper parameter $A_1 \neq 0$ and one lower parameter $B_1 \neq 1$, the β_j again all vanish, and a parametrization $z \rightarrow \xi(z)$ should exist such that $[(1-z)h(z)]^{-1} dz/d\xi$, $[zh(z)]^{-1} dz/d\xi$, $z^{-1} dz/d\xi$, are rational functions of ξ , where $h(z) = C z^{1-B_1} (z-1)^{B_1-A_1-1}$. (See Ref. [24].)

Example III: Let us consider a Gauss hypergeometric function of the form $\omega(z) = {}_2F_1(\frac{p_1}{q} + a_1\varepsilon, \frac{p_2}{q} + a_2\varepsilon; 1 - \frac{r}{q} + c\varepsilon; z)$, where p_1, p_2, q, r are integers. It is a solution of the differential equation

$$\left(z \frac{d}{dz} + \frac{p_1}{q} + a_1\varepsilon \right) \left(z \frac{d}{dz} + \frac{p_2}{q} + a_2\varepsilon \right) \omega(z) = \frac{d}{dz} \left(z \frac{d}{dz} - \frac{r}{q} + c\varepsilon \right) \omega(z) \quad (41)$$

for coefficient functions $\omega_k(z)$ defined via the expansion $\omega(z) = 1 + \sum_{k=1}^{\infty} \omega_k(z) \varepsilon^k$. Eq. (41) produces an infinite system of linear differential equations

$$\begin{aligned} & \left[(1-z) \frac{d}{dz} + \left(\beta - \frac{p_1+p_2}{q} \right) - \frac{1}{z} \left(\beta + \frac{r}{q} \right) \right] \left(z \frac{d}{dz} + \beta \right) \omega_k - \left[\left(\beta - \frac{p_1}{q} \right) \left(\beta - \frac{p_2}{q} \right) - \frac{1}{z} \beta \left(\beta + \frac{r}{q} \right) \right] \omega_k(z) \\ &= \left(a_1 + a_2 - \frac{c}{z} \right) \left(z \frac{d}{dz} + \beta \right) \omega_{k-1}(z) + \frac{c}{z} \beta \omega_{k-1} - \left[a_1 \left(\beta - \frac{p_2}{q} \right) + a_2 \left(\beta - \frac{p_1}{q} \right) \right] \omega_{k-1}(z) + a_1 a_2 \omega_{k-2}(z), \end{aligned} \quad (42)$$

where β is arbitrary and k runs from 0 to ∞ . The second order differential equation (42) can be split into two first-order differential equations by introducing new functions

$$\rho_k(z) = \left(z \frac{d}{dz} + \beta \right) \omega_k(z) , \quad (43)$$

so that

$$\begin{aligned} & \left[(1-z) \frac{d}{dz} + \left(\beta - \frac{p_1+p_2}{q} \right) - \frac{1}{z} \left(\beta + \frac{r}{q} \right) \right] \rho_k(z) - \left[\left(\beta - \frac{p_1}{q} \right) \left(\beta - \frac{p_2}{q} \right) - \frac{1}{z} \beta \left(\beta + \frac{r}{q} \right) \right] \omega_k(z) \\ &= \left(a_1 + a_2 - \frac{c}{z} \right) \rho_{k-1}(z) + \frac{c}{z} \beta \omega_{k-1} - a_1 \left(\beta - \frac{p_2}{q} \right) \omega_{k-1}(z) - a_2 \left(\beta - \frac{p_1}{q} \right) \omega_{k-1}(z) + a_1 a_2 \omega_{k-2}(z). \end{aligned} \quad (44)$$

We can use the fact that this system takes triangular form when the last term in the l.h.s. of the second equation is zero to obtain the following solutions:

$$p_1 p_2 = 0 \longrightarrow \beta = 0 , \quad (45)$$

$$p_1 = 0 \longrightarrow \beta = 0 , \quad (46)$$

$$\beta = -\frac{r}{q} = \frac{p_1}{q} , \quad p_2 = p . \quad (47)$$

We would like to analyze the case of the parameters in Eq. (47) in detail. Then

$$\rho_k(z) = \left(z \frac{d}{dz} - \frac{r}{q} \right) \omega_k(z), \quad (48)$$

$$\left[(1-z) \frac{d}{dz} - \frac{p}{q} \right] \rho_k(z) = \left(a_1 + a_2 - \frac{c}{z} \right) \rho_{k-1}(z) + \frac{c}{z} \frac{p_1}{q} \omega_{k-1}(z) - a_1 \frac{p_1 - p}{q} \omega_{k-1}(z) + a_1 a_2 \omega_{k-2}(z). \quad (49)$$

The redefinition $(\omega_k, \rho_k) \rightarrow (z^{\frac{r}{q}} \pi_k, (1-z)^{-\frac{p}{q}} \sigma_k)$ leads to a new set of equations

$$\sigma_k(z) = h(z) z \frac{d}{dz} \pi_k(z), \quad (50)$$

$$(1-z) \frac{d}{dz} \sigma_k(z) = \left(a_1 + a_2 - \frac{c}{z} \right) \sigma_{k-1}(z) + \frac{c}{z} \frac{p_1}{q} h(z) \pi_{k-1}(z) - a_1 \frac{p_1 - p}{q} h(z) \pi_{k-1}(z) + a_1 a_2 h(z) \pi_{k-2}(z), \quad (51)$$

where $h(z) = (1-z)^{p/q} z^{r/q}$. The result can be expressed in terms of Goncharov's polylogarithms if there is a parametrization $z \rightarrow \xi(z)$ such that $[zh(z)]^{-1} dz/d\xi$, $[(1-z)h(z)]^{-1} dz/d\xi$, $z^{-1} dz/d\xi$, and $(1-z)^{-1} dz/d\xi$ are rational functions. (See Ref. [24].) Such a parametrization exists when $p = -r$ and $z \rightarrow \xi = \left(\frac{z}{z-1} \right)^{1/q}$. In this way, we find the following set of conditions:

$$\beta = -\frac{r}{q} = \frac{p_1}{q} = \frac{p_2}{q}. \quad (52)$$

Under these conditions, the coefficients of the ε -expansion of the function ${}_2F_1\left(\frac{p_1}{q} + a_1\varepsilon, \frac{p_2}{q} + a_2\varepsilon; 1 - \frac{r}{q} + c\varepsilon; z\right)$, are expressible in term of Goncharov's polylogarithms. Eq. (52) corresponds to Lemma IV of Ref. [23], but the present derivation does not rely on any symmetries of Gauss hypergeometric functions.

Remark 1: For a Gauss hypergeometric function, the $\varepsilon = 0$ term should be a rational (not just algebraic) function.

Remark 2: For Eq. (46), an additional relation between p_2 and r arises from Eq. (38), and we arrive again at one of the cases of Eq. (40).

Let us consider the ε expansion of a hypergeometric function with the following set of parameters: $\omega(z) = {}_pF_{p-1}(\mathbf{a}\varepsilon, A_1 + c_1\varepsilon, A_2 + c_2\varepsilon; \mathbf{b}\varepsilon, B_1 + f_1\varepsilon, B_2 + f_2\varepsilon; z)$, where $A_1, A_2, B_1, B_2, \mathbf{a}, \mathbf{b}, c$, and f are rational numbers. Eqs. (37) take the form

$$\begin{aligned} R_2 + \beta &= (B_1 - 1) + (B_2 - 1), & R_1 + \beta &= A_1 + A_2, \\ R_2\beta &= (B_1 - 1)(B_2 - 1), & R_1\beta &= A_1 A_2. \end{aligned} \quad (53)$$

Eq. (40) provides additional conditions on the relations between R_1 and R_2 , and as a consequence, there are three different solutions:

$$R_1 = R_2 : \quad B_1 - 1 + B_2 - 1 = A_1 + A_2, \quad (B_1 - 1)(B_2 - 1) = A_1 A_2; \quad (54)$$

$$\begin{aligned} R_1 = 0 : \quad & A_2 = 0, \quad \beta = A_1, \\ & R_2 = (B_1 - 1) + (B_2 - 1) - A_1, \quad R_2 A_1 = (B_1 - 1)(B_2 - 1); \end{aligned} \quad (55)$$

$$\begin{aligned} R_2 = 0 : \quad & B_2 = 1, \quad \beta = B_1 - 1 \\ & R_1 = A_1 + A_2 - (B_1 - 1), \quad R_1(B_1 - 1) = A_1 A_2; \end{aligned} \quad (56)$$

The solutions of Eqs. (54) and (56) are the roots of a quadratic equation $x^2 - (A_1 + A_2)x + A_1 A_2 = 0$, and the solution of Eq. (55) satisfies the same quadratic equation with the replacement $A_i \rightarrow B_i - 1$. One solution of Eq. (54) is $B_j = 1 + A_j$, and one solution of Eqs. (55) and (56) is $B_1 = 1 + A_1$.

There is another parametrization for the same hypergeometric function. Let us rewrite the operator $D_0^{(p)}$ in

Eq. (35) as

$$\begin{aligned} & \left\{ \left[-(1-z) \frac{d}{dz} + \sum_{k=1}^2 A_k - \frac{1}{z} \sum_{j=1}^2 (B_j - 1) \right] \theta + A_1 A_2 - \frac{1}{z} (B_1 - 1)(B_2 - 1) \right\} \theta^{p-2} \\ &= \left\{ \left[-(1-z) \frac{d}{dz} - (\beta - A_1 - A_2) + \frac{1}{z} (\beta - B_1 - B_2 + 2) \right] (\theta + \beta) \right\} \theta^{p-2} \\ &+ \left\{ (\beta - A_1)(\beta - A_2) - \frac{1}{z} (\beta - B_1 + 1)(\beta - B_2 + 1) \right\} \theta^{p-2}. \end{aligned} \quad (57)$$

The first condition is that there should exist a common factor for the last line in Eq. (57), for example, $B_1 = A_1 + 1$, and consequently, $\beta = A_1$.

Example IV: Let us consider the ε expansion of a hypergeometric function ${}_3F_2$ with the following set of parameters: $\omega(z) = {}_3F_2\left(\frac{r}{q} + a_1\varepsilon, a_2\varepsilon, a_3\varepsilon; 1 + \frac{r}{q} + b_1\varepsilon, 1 - \frac{p}{q} + b_2\varepsilon; z\right)$ with $a_1, a_2, a_3 \neq 0$. Starting from the differential equation for this hypergeometric function,

$$\left[z \left(\theta + \frac{r}{q} + a_1\varepsilon \right) (\theta + a_2\varepsilon) (\theta + a_3\varepsilon) - \theta \left(\theta + \frac{r}{q} + b_1\varepsilon \right) \left(\theta - \frac{p}{q} + b_2\varepsilon \right) \right] \omega(z) = 0. \quad (58)$$

the system of differential equations for the coefficient functions $w_k(z)$ in its ε expansion $\omega(z) = 1 + \sum_{j=1}^{\infty} w_j(z) \varepsilon^j$ is found to be

$$\begin{aligned} & \left[(1-z) \frac{d}{dz} - \frac{1}{z} \frac{p}{q} \right] \left(\theta + \frac{r}{q} \right) \theta \omega_m(z) = \left[a_1 + a_2 + a_3 - \frac{b_1 + b_2}{z} \right] \left(\theta + \frac{r}{q} \right) \theta \omega_{m-1}(z) + a_2 a_3 \frac{r}{q} \omega_{m-2}(z) \\ & + a_1 a_2 a_3 \omega_{m-3}(z) + \delta_1 \theta \omega_{m-1}(z) + \delta_2 \theta \omega_{m-2}(z) + \frac{1}{z} \delta_3 \theta \omega_{m-1}(z) + \frac{1}{z} \delta_4 \theta \omega_{m-2}(z). \end{aligned} \quad (59)$$

where $\theta = z d/dz$ and δ_j are constants. After a redefinition

$$\left(\omega_k(z), \theta \omega_k(z), \left(\theta + \frac{r}{q} \right) \theta \omega_k(z), \right) \rightarrow \left(\omega_k(z), z^{-r/q} \sigma_k(z), \left(\frac{z}{z-1} \right)^{p/q} \phi_k(z) \right), \quad (60)$$

we find that the function $h(z)$ defined by (39) is $h(z) = z^{(p+r)/q} (z-1)^{-p/q}$. By (40), a rational parametrization is possible when $p = -r$.

The methods described here can be extended to any multi-loop Horn-type hypergeometric function. Starting from the Pfaff form of the differential equation,

$$d\phi^{(i)}(z, \varepsilon) = \sum_{k,j} P_{i,j,k}(z, \varepsilon) \phi^{(j)}(z, \varepsilon) dz_k, \quad (61)$$

where $P_{i,j,k}(z, \varepsilon)$ are rational functions, the system can be transformed to triangular form and integrated.

Let us consider the ε -expansion of the Appell hypergeometric function F_3 , which was analyzed in the context of photon box diagrams [63]:

$$\begin{aligned} & F_3 \left(\frac{p_1}{q} + a_1\varepsilon, \frac{p_2}{q} + a_2\varepsilon, \frac{r_1}{q} + b_1\varepsilon, \frac{r_2}{q} + b_2\varepsilon, 1 - \frac{p}{q} + c\varepsilon; x, y \right) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{p_1}{q} + a_1\varepsilon \right)_m \left(\frac{p_2}{q} + a_2\varepsilon \right)_n \left(\frac{r_1}{q} + b_1\varepsilon \right)_m \left(\frac{r_2}{q} + b_2\varepsilon \right)_n}{\left(1 - \frac{p}{q} + c\varepsilon \right)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}. \end{aligned} \quad (62)$$

Applying our methods, we find that the coefficients of ε -expansion may be expressed in terms of Goncharov's polylogarithms when $p_j r_j = 0$ for $j = 1, 2$, and a rational parametrization should exist for the functions

$$h_1(x) = (-1)^{s_1/q} \left[\frac{x^p}{(x-1)^{s_1+p}} \right]^{1/q}, \quad h_2(x) = (-1)^{s_2/q} \left[\frac{y^p}{(y-1)^{s_2+p}} \right]^{1/q}, \quad (63)$$

$$H(x, y) = (-1)^{(s_1+s_2)/q} \left[\frac{x^{s_2+p} y^{s_1+p}}{(xy - x - y)^{s_1+s_2+p}} \right]^{1/q}, \quad (64)$$

where $s_j = p_j + r_j$ and $j = 1, 2$. As result of our analysis, we claim that in two cases (only),

$$F_3 \left(I_1 + \frac{p_1}{q} + a_1\varepsilon, I_2 + a_2\varepsilon, I_3 + b_1\varepsilon, I_4 + b_2\varepsilon, I_5 + \frac{p_1}{q} + c\varepsilon; x, y \right), \quad (65)$$

$$F_3 \left(I_1 + \frac{p_1}{q} + a_1\varepsilon, I_2 + a_2\varepsilon, I_3 + b_1\varepsilon, I_4 + b_2\varepsilon, I_5 + c\varepsilon; x, y \right), \quad (66)$$

with integer values I_j, p_1, q , the ε expansion F_3 can be expressed in terms of Goncharov polylogarithms. We note that for F_3 , the $\varepsilon = 0$ term should be a rational function.

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